Revisiting the Ontological Square

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Abstract. Considerations regarding predication in ordinary language as well as the ontology of relations suggest a refinement of the Ontological Square, a conceptual scheme used in many foundational ontologies and which consists of particular substrates as well as their types on the one hand and particular attributes as well as their types on the other hand. First, the distinction between particulars and universals turns out to be one of degree, since particulars are merely the least elements in the subsumption hierarchy. Second, relations may be analysed in terms of roles as ways of participating in events. In consequence, the Logic of the Ontological Square proposed in [1,2] has to be revised accordingly.

Keywords. Universals, particulars, substrates, attributes, relations, predication, attribution, subsumption, foundational ontologies, higher-order logic

1. Introduction

1.1. Motivation

In my [1,2] I discuss a four-categorial scheme that is recurrent in foundational ontologies, namely the so-called Ontological Square. This scheme consists of particular substrates as well as their types on the one hand and particular attributes (features) as well as their types on the other hand. For the purposes of this study it is sufficient to think of substrates as any bearers of features, be they qualities, realisable traits such as dispositions, functions or roles, or relations. I would like to remain neutral as to whether (particular) substrates are perdurants or endurants, i.e. whether they have temporal parts or not.[2]

This paper revisits the Ontological Square as well as my prior attempts to provide a logical calculus that is based on the intuitions underpinning Aristotelian substrate-attribute ontology. I will start with considerations on predication in ordinary language which are mainly due to Jonathan Lowe [4,5]. Though these considerations support an Aristotelian view of ontology, they also suggest that the distinction between particulars and universals boils down to a difference of degree: particulars as their own infimae species are merely the lowest elements in the subsumption hierarchy. A further major difference with the assays of four-category ontology in [1,2] is a neo-Aristotelian analysis

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[2]Some authors, such as McCall and Lowe [3], actually argue that endurantism and perdurantism are equivalent accounts of persistence through time and change.

of relations in terms of relational attributes or roles, more precisely in terms of roles as ways of participating in events.

All this considerations suggest a considerable revision of the Logic of the Ontological Square (LoOS), a system of higher-order logic proposed in [1,2], a revision which will be described in the second part of this paper.

1.2. Subsumption, attribution and exemplification

According to the conventional view of predication, an elementary proposition consists in the application of a predicate expressing a universal attribute to one or several terms that denote particular substrates. The underlying ontology, which Smith [6] calls *fantology*, but could also be appropriately named the *Ontological Diagonal*, is a bi-categorial scheme that combines repeatable attributes with non-repeatable substrates :

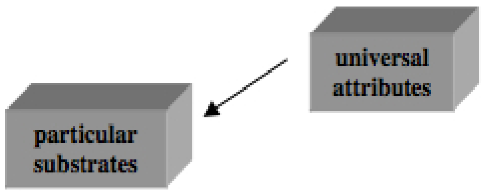

Figure 1. The Ontological Diagonal

However, E. J. Lowe [4] rightly points out that predication in natural languages is richer than the one in classical predicate logic. Indeed, we need to distinguish between the copula of subsumption and the copula of attribution. The copula of subsumption ("is a" or "are") combines terms denoting particulars or universals with terms denoting universals:

- This (particular organism) is a bacterium.
- Staphylococci are bacteria.

Ordinary language and traditional logic do not differentiate between cases in which the subsumee refers to a particular and those cases in which the subsumee denotes a universal. Thus, it seems arbitrary to draw a distinction between the instantiation of universals by particulars and the subsumption of universals by other universals.

The copula of attribution ("is" or "are") also combines terms denoting particulars or universals with terms referring to universals:

- This (particular organism) is unicellular.
- Bacteria are unicellular.

Contrary to the conventional view, ordinary language and traditional logic allow for universal predicates applying to terms denoting universals. This generic attribution is not to be confused with second-order predication: it pertains to generic features of kinds or types of substrates.

At this point it becomes obvious that it is erroneous to conflate the distinction between substrates and attributes with that between particulars and universals. Indeed, there seem to be two sorts of general terms: those referring to universal substrates or kinds and those denoting universal attributes. It seems convenient to distinguish between generic attribution and what should be properly called *exemplification*, which holds between attribute universals and particular substrates.

Now, the use of aspects in English may indicate that "exemplification" actually covers two distinct ties between universal attributes and particular substrates:

1. This (portion of) gold melts at 1064,18°C.
2. This (portion of) gold is melting at 1064,18°C.

Indeed, while the present simple in (1) expresses a *dispositional* exemplification, the present continuous in (2) suggests an *occurrent* exemplification [5]. (1) affirms that a certain portion of gold has the disposition to melt at 1064,18°C, while (2) describes the actual occurrence of this portion of gold melting at 1064,18°C. Lowe (ibid.) suggests that the distinction between dispositional and occurrent exemplification can be explained, if one assumes that besides generic attribution on the level of universals, there is also specific attribution on the level of particulars. Thus, *dispositional* exemplification holds between a particular substrate and a universal attribute iff the latter is a generic attribute of a substantial universal which is instantiated by the former. E.g. since Melting-at-1064,18°C is an attribute of Gold, the portion under investigation dispositionally exemplifies Melting-at-1064,18°C, even without presently doing so. *Occurrent* exemplification holds between a particular substrate and a universal attribute iff the former has a trope that instantiates the latter. E.g. our portion "occurrently" exemplifies Melting-at-1064,18°C, iff it is the substrate of a particular instance of melting at 1064,18°C.

1.3. A four-category ontology

The considerations above suggest a four-fold classification of the denizens of reality, namely into particular substrates (*objects*) as well as universal substrates (*kinds*) on the one hand and particular attributes (*tropes*) as well as universal attributes (*properties*) on the other hand. This categorial scheme, which has been read into Aristotle's *Categories* (more precisely lines 1a20–1b10) by Ackrill [7], Villemin [9] and Angelelli [8], is commonly referred to as the "Ontological Square":

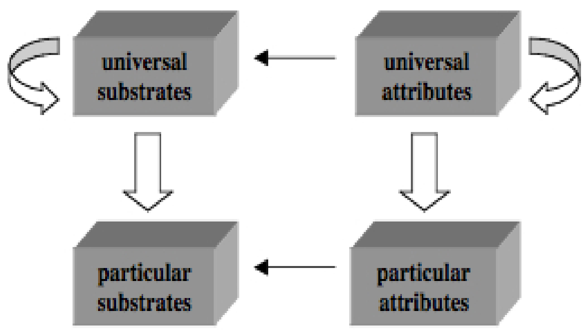

Figure 2. The Ontological Square

The Ontological Square is structured by two ontological ties or nexus, namely *subsumption* (indicated by thick arrows in Figure 2) and *attribution* (indicated by thin arrows in Figure 2). This Aristotelian four-category ontology has recently been defended by Barry Smith [10,6], Jonathan Lowe [5] and myself [1,2]; as such it can be said to underly two major foundational ontologies, namely DOLCE [11] and BFO [12].

In the version of the Ontological Square described above, subsumption applies to particulars and universals indifferently. Thus, each particular turns out to be its own infima species. This view has not only been shared by Leibniz [13][3], but is also akin to the assimilation of set-theoretical *urelements* to their singleton sets in Quine's system ML [14, par. 25].

Given that each particular or *instance* is its own infima species, there are no "bare" particulars the only *raison d'être* of which is to be bearers of their attributes (*pace* Bergmann [15]). Indeed, all particulars, whether objects or tropes, are – as infimae species of themselves – members of the subsumption hierarchy just like universals. Subsumption, however, is a tie between quiddities, i.e. essences. Thus, the distinction between an entity and its essence is only notional.

Another consequence is that the universal-particular distinction is not an absolute divide, as has been pointed out by MacBride [16], but only a matter of degree: particulars are merely the least elements in the subsumption hierarchy. Of course, this does not mean that this dichotomy is vacuous, as Ramsey [17] has contended. Indeed, in order to account for the difference between occurrent and dispositional exemplification it is mandatory to distinguish generic attribution between universals and specific attribution between particulars. Furthermore, attribution between particulars and that between universals differ inasmuch as particular attributes or tropes are bearer-specific, i.e. each trope, e.g. a particular redness, is the attribute of exactly one object, e.g. a particular rose.

It is often maintained that ties such as attribution or subsumption are no addition to being, that they are grounded by their terms [18, p. 12]. There is no room here to discuss this stance in detail, but suffice it to point out that if the terms of a tie were bare particulars, the thesis in question would entail that subsumption and attribution hold of their terms as a brute fact. The view that ties supervene on their terms is nonetheless much less counterintuitive if the terms are quiddities. Now, in the ontological framework presented here, there are no bare particulars, since all entities, universals and instances alike, are quiddities. The ties of subsumption and attribution may then be explained in terms of affinities between essences, just like chemical bonds in molecules are based on the valences of their constituent atoms.

1.4. Relations in the Ontological Square

In [1,2], I have taken the standard account of relations as being fundamentally characterised by order and arity simply for granted. However, Kit Fine [19, pp. 2 ff] forcefully argues that this account according to which relations apply to their relata in a specific order is flawed. In order to show this, one just needs to appeal to the trivial fact that each binary relation, e.g. *Loves*, has its converse, e.g. *Loved-By*. Now, the assertion that John loves Mary is true in virtue of John loving Mary or Mary being loved by John. It seems plausible that John's loving Mary and Mary's being loved by John are descriptions of the same states of affairs. But could the same states of affairs be constituted by two distinct relations ? It would seem not. But then, which one of the two relations is the more basic, in the sense of being constitutive of the states of affairs, and which one is derived ?

[3] "...ce que S. Thomas asseure sur ce point des anges ou intelligences (quod ibi omne individuum sit species infima) est vray de toutes les substances..." [13, chap. ix]

This conundrum seems to have a plausible solution, namely positionalism, i.e. the view that relations do not entail any specific order, but have positions or argument-places to be filled by their terms. Thus the Loving-Relation would have two positions: one for the Lover and one for the Beloved [19, pp. 10 ff]. But positionalism seems to fails to account for relations which obviously do not have qualitatively distinct argument-places, so-called *neutral relations* such as the Sibling-relation [19, pp. 17 ff].

Now, it seems to me that, *pace* Fine, there is a variant of positionalism which even works for neutral relations. Indeed, assume that positions are actually particulars, namely *tropes*: thus, in the case of neutral relations, they may be qualitatively identical, but nonetheless quantitatively distinct. Of course, this would presuppose that we reify relations and view them as objects with respect to which other objects can have roles. Relations would turn out to be tantamount to *occurrents* in which other objects, including other occurrents, may participate in. Such an account is akin to the event-based analysis of predication in natural language as it has been proposed by Davidson [20] and developed by Parsons [21].

Therefore, besides attribution, there is yet another tie between attributes and substrates, namely the tie of being a role in some substrate, which I will call *relation*. Since there are role-universals such as *Agent* or *Recipient*, there is a generic as well as a specific form of relation: e.g. the occurrent-kind *Loving* has three generic roles, namely *Experiencer* and *Patient* or *Theme*. Thus roles, whether universals or particulars, are basically extrinsic attributes or relational attributes, which is, after all, perfectly Aristotelian.

The remainder of this article presents the revised Logic of the Ontological Square, LoOS^{+}, its axiom system as well as its metatheory. As detailed in [2], LoOS^{+} is a copula logic rather than a predicate logic, copulae being uninterpreted operators that form sentences out of terms denoting substrates and attributes alike. Hence a copula calculus is distinguished from Fregean predicate calculus by the fact that categoremata or lexemes and syncategoremata or grammemes are strictly separated.

2. The formal system LoOS^{+}

2.1. The Language of LoOS^{+}

2.1.1. Signature of LoOS^{+}

LoOS^{+} contains four sorts of variables, respectively constants:

- *kind variables*: $x_k, y_k, z_k, x_k{}', y_k{}', z_k{}', x_k{}'', y_k{}'', z_k{}'', \ldots$
- *kind constants*: $a_k, b_k, c_k, a_k{}', b_k{}', c_k{}', a_k{}'', b_k{}'', c_k{}'', \ldots$
- *property variables*: $x_p, y_p, z_p, x_p{}', y_p{}', z_p{}', x_p{}'', y_p{}'', z_p{}'', \ldots$
- *property constants*: $a_p, b_p, c_p, a_p{}', b_p{}', c_p{}', a_p{}'', b_p{}'', c_p{}'', \ldots$
- *object variables*: $x_o, y_o, z_o, x_o{}', y_o{}', z_o{}', x_o{}'', y_o{}'', z_o{}'', \ldots$
- *object constants*: $a_o, b_o, c_o, a_o{}', b_o{}', c_o{}', a_o{}'', b_o{}'', c_o{}'', \ldots$
- *trope variables*: $x_t, y_t, z_t, x_t{}', y_t{}', z_t{}', x_t{}'', y_t{}'', z_t{}'', \ldots$
- *trope constants*: $a_t, b_t, c_t, a_t{}', b_t{}', c_t{}', a_t{}'', b_t{}'', c_t{}'', \ldots$

Kind and property terms are referred to as *universal terms*, while object and trope terms are called *instance terms*. Furthermore, kind and object terms are referred to as *substrate terms*, while property and trope terms are called *attribute terms*.

Sans serif letters (with or without subscripts or superscripts) are used as schematic variables or constants in the meta-language. The letters t, t', t'', etc. (with or without subscripts or superscripts) stand for any terms. In particular, x_i, $\mathsf{x}_i{}'$, $\mathsf{x}_i{}''$, etc., a_i, $\mathsf{a}_i{}'$, $\mathsf{a}_i{}''$, etc., t_i, $\mathsf{t}_i{}'$, $\mathsf{t}_i{}''$, etc. respectively stand for variables, constants and terms of the sort i.

LoOS$^+$ contains seven formal predicates or copulae:

1. subsumption between substrates: "SB_s" (read: "subsumes"),
2. subsumption between attributes: "SB_a" (read: "subsumes")),
3. attribution between universals: "AT_u" (read: "(is an) attribute of"),
4. attribution between instances: "AT_i" (read: "(is an) attribute of"),
5. relation between universals: "RE_u" (read: "(is a) role in") ,
6. relation between instances: "RE_i" (read: "(is a) role in"),
7. identity ("$=$").

Functional terms can be introduced as definite descriptions, which in turn can be introduced by implicit definition in the usual manner (see below).

2.1.2. Well-formed formulae of LoOS$^+$

The logical constants of LoOS$^+$ are negation ($\neg$), implication ($\rightarrow$) and the universal quantifier ($\forall\mathsf{x}$); brackets are used to delimit the scope of these operators. All other logical constants (conjunction $\wedge$, disjunction $\vee$, equivalence $\leftrightarrow$, and the existential quantifier $\exists\mathsf{x}$) are defined. The lower-case greek letters ϕ, ψ and ξ are used as schematic variables for formulae. Well-formed formulae (wff) of LoOS$^+$ are recursively defined as follows:

1. if t is a kind term and t' any substrate term, then $\ulcorner\mathsf{t}\,\mathsf{SB}_s\mathsf{t}'\urcorner$ is a wff;
2. if t is a property term and t' any attribute term, then $\ulcorner\mathsf{t}\,\mathsf{SB}_a\mathsf{t}'\urcorner$ is a wff;
3. if t is a property term and t' a kind term term, then $\ulcorner\mathsf{t}\,\mathsf{AT}_u\mathsf{t}'\urcorner$ is a wff;
4. if t is a trope term and t' an object term, then $\ulcorner\mathsf{t}\,\mathsf{AT}_i\mathsf{t}'\urcorner$ is a wff;
5. if t is a property term and t' a kind term, then $\ulcorner\mathsf{t}\,\mathsf{RE}_u\mathsf{t}'\urcorner$ is a wff;
6. if t is a trope term and t' an object term, then $\ulcorner\mathsf{t}\,\mathsf{RE}_i\mathsf{t}'\urcorner$ is a wff;
7. if t_i and t_i' are terms of the same sort i, then $\ulcorner\mathsf{t}_i = \mathsf{t}_i'\urcorner$ is a wff;
8. if ϕ and ψ are wffs, then so are $\neg(\phi)$ and $(\phi \rightarrow \psi)$;
9. If ϕ is a wff and x is any variable, then $\ulcorner\forall\mathsf{x}(\phi)\urcorner$ is a wff.

I shall omit parentheses where this does not lead to confusion. In particular formulae like $\ulcorner\forall\mathsf{x}(\forall\mathsf{y}(\phi))\urcorner$ will be shortened to $\ulcorner\forall\mathsf{x}\forall\mathsf{y}(\phi)\urcorner$.

If x is a variable occurring in a well-formed formula ϕ, then x is *bound* by $\forall\mathsf{x}$ within $\forall\mathsf{x}(\phi)$, except in subformulae of the form $\forall\mathsf{x}(\psi)$. Any variable x that is not bound in a formula ϕ is *free in* ϕ. A term t is *free in* ϕ iff t is a constant or a variable that is free in ϕ. in A formula ϕ with no free variables is called a *closed formula* or a *sentence*. If ϕ is any formula, and t_1 a term occurring free in ϕ, then $\phi^{\mathsf{t}_2}_{\mathsf{t}_1}$ is the result of substituting one or more occurrences of t_1 in ϕ with occurrences of t_2.

Definite descriptions can be defined as a special sort of quantifiers [22, p. 173], i.e., for any sort i, any formula ϕ containing the sole free variable x_i:

Definition 1. $[\iota\mathsf{x}_i\phi]\psi(\iota\mathsf{x}_i\phi) \equiv_{df} \exists\mathsf{y}_i\forall\mathsf{z}_i\,((\phi^{\mathsf{z}_i}_{\mathsf{x}_i} \leftrightarrow \mathsf{z}_i = \mathsf{y}_i) \wedge \psi(\mathsf{y}_i))$ [4]

[4]This definition has the apparent flaw that it may be the case that $\ulcorner[\iota\upsilon\phi(\upsilon)]\psi(\iota\upsilon\phi(\upsilon))\urcorner$ is false, while $\ulcorner\forall\upsilon\psi(\upsilon)\urcorner$ is true. However, no inconsistencies arise, since the axioms and rules for the quantifiers operate on the level of the basic language where all defined terms are resolved into their defining terms.

2.2. The Deductive System of LoOS$^+$

2.2.1. Logical axioms and inference rules

The logical axioms and inference rules of LoOS$^+$ are those of standard predicate logic with identity. For any formulae ϕ, ψ or ξ, the axioms of LoOS$^+$ are as follows:

Axiom 1. $\phi \rightarrow (\psi \rightarrow \phi)$

Axiom 2. $(\phi \rightarrow (\psi \rightarrow \xi)) \rightarrow ((\phi \rightarrow \psi) \rightarrow (\phi \rightarrow \xi))$

Axiom 3. $(\neg\phi \rightarrow \neg\psi) \rightarrow (\psi \rightarrow \phi)$

Axiom 4. (For any sort i, for any formula ϕ, variable x_i and term t_i occurring free in ϕ)

$\forall \mathsf{x}_i\, \phi(\mathsf{x}_i) \rightarrow \phi(\mathsf{t}_i)$

Axiom 5. $\forall \mathsf{x}\, (\mathsf{x} = \mathsf{x})$

Axiom 6. $\forall \mathsf{x}\, \forall \mathsf{y}\, (\, \mathsf{x} = \mathsf{y} \;\rightarrow\; (\, \phi \leftrightarrow \phi^{\mathsf{y}}_{\mathsf{x}}\,)\,)$

A formula ϕ is *derivable in LoOS$^+$* from a set of formulae Δ ($\Delta \vdash_{LoOS^+} \phi$) iff there is a finite sequence of formulae $\phi_1, \ldots, \phi_n$ such that $\phi = \phi_n$ and, for any $i \leq n$, ϕ_i is in Δ, and is either an axiom of LoOS$^+$ or such as to follow from previous members of the sequence by one of the following inference rules.

Rule 1. (For any formulae ϕ and ψ:) From ϕ, $\phi \rightarrow \psi$ infer ψ

Rule 2. (For any formulae ϕ and ψ and any variable x that does not occur free in ϕ or in any of the premisses in the derivation:)

From $\phi \rightarrow \psi(\mathsf{x})$ infer $\phi \rightarrow \forall \mathsf{x}\, \psi(\mathsf{x})$

2.2.2. Axioms of predication

The axioms specific to LoOS$^+$ are those that pertain to the six copulae. I adopt a minimal characterisation of the different modes of predication, in particular I deliberately leave out any existence claims. The latter already belong to metaphysics proper and are thus beyond the scope of formal logic as formal ontology.

Subsumption between substrates resp. attributes is antisymmetric and transitive.[5]

Axiom 7. $\forall \mathsf{x}\, \forall \mathsf{y}\, (\, (\mathsf{x}\, \mathsf{SB}_s \mathsf{y} \wedge \mathsf{y}\, \mathsf{SB}_s \mathsf{x}\,) \rightarrow \mathsf{x} = \mathsf{y}\,)$

Axiom 8. $\forall \mathsf{x}\, \forall \mathsf{y}\, \forall \mathsf{z}\, (\, (\, \mathsf{x}\, \mathsf{SB}_s \mathsf{y} \;\wedge \mathsf{y}\, \mathsf{SB}_s \mathsf{z}) \rightarrow \mathsf{x} \mathsf{SB}_s \mathsf{z}\,)$

Axiom 9. $\forall \mathsf{x}\, \forall \mathsf{y}\, (\, (\mathsf{x}\, \mathsf{SB}_a \mathsf{y} \wedge \mathsf{y}\, \mathsf{SB}_a \mathsf{x}\,) \rightarrow \mathsf{x} = \mathsf{y}\,)$

Axiom 10. $\forall \mathsf{x}\, \forall \mathsf{y}\, \forall \mathsf{z}\, (\, (\, \mathsf{x}\, \mathsf{SB}_a \mathsf{y} \;\wedge \mathsf{y}\, \mathsf{SB}_a \mathsf{z}) \rightarrow \mathsf{x} \mathsf{SB}_s \mathsf{z}\,)$

[5] The formation rules are enough to ensure that the terms on both sides of the respective copula are of the right sort.

Properties are inherited downwards the subsumption hierarchy. In other words, properties of the subsumer should also be properties of the subsumee. Note that this axiom is meant to act as a constraint both on attribution (between universals) and on subsumption (between kinds): in case of doubt, either the attribute under consideration is not a property of the subsumer or subsumption does not hold.

Axiom 11. $\forall \mathsf{x} \forall \mathsf{y} \forall \mathsf{z}\,(\,(\,\mathsf{x}\,\mathsf{AT}_u \mathsf{y}\ \wedge \mathsf{y}\,\mathsf{SB}_s \mathsf{z}) \rightarrow \mathsf{x}\mathsf{AT}_u \mathsf{z}\,)$

A trope is an attribute of at most one object.

Axiom 12. $\forall \mathsf{x} \forall \mathsf{y} \forall \mathsf{z}\,(\,(\,\mathsf{x}\,\mathsf{AT}_i \mathsf{y}\ \wedge \mathsf{x}\,\mathsf{AT}_i \mathsf{z}\,) \rightarrow \mathsf{y} = \mathsf{z}\,)$

Just like properties, roles are inherited downwards the subsumption hierarchy. In other words, roles in the subsumer should also be roles in the subsumee. Again, this axiom is meant to act as a constraint both on relation (between universals) and on subsumption (between kinds): in case of doubt, either the attribute under consideration is not a role in the subsumer or subsumption does not hold.

Axiom 13. $\forall \mathsf{x} \forall \mathsf{y} \forall \mathsf{z}\,(\,(\,\mathsf{x}\,\mathsf{RE}_u \mathsf{y}\ \wedge \mathsf{y}\,\mathsf{SB}_s \mathsf{z}) \rightarrow \mathsf{x}\mathsf{RE}_u \mathsf{z}\,)$

A trope is a role in at most one object.

Axiom 14. $\forall \mathsf{x} \forall \mathsf{y} \forall \mathsf{z}\,(\,(\,\mathsf{x}\,\mathsf{RE}_i \mathsf{y}\ \wedge \mathsf{x}\,\mathsf{RE}_i \mathsf{z}\,) \rightarrow \mathsf{y} = \mathsf{z}\,)$

Given the parallelism between the axioms for attribution and relation, I should point out that I leave it open whether a substrate can have an attribute that is a role with respect to the same substrate. (Universals) of self-regulatory processes come immediately to mind as an example, since it would seem that the latter have the role of regulators with respect to themselves.

Note that every axiom of predication is the universal quantification of a conditional with a conjunction of atomic formulae in the antecedent and an atomic formula in the consequent. This means that each axiom could be replaced by an inference rule having the conjuncts of the antecedent as premisses and the consequent as the conclusion.

3. Meta-theory of LoOS$^+$

3.1. A model for LoOS$^+$

A LoOS$^+$ model is akin to a first-oder model for standard second-order logic [24]. Indeed, a model $\mathfrak{M}$ of LoOS$^+$ is a quadruple $\langle \mathfrak{U}, \mathfrak{u}, \mathfrak{p}, \mathfrak{I} \rangle$ such that:

1. $\mathfrak{U}$ is a is a non-empty set called *universe of entities*
2. $\mathfrak{u}$ is a function from the interval $[0, 7]$ into the powerset of $\mathfrak{U}$ such that:
   (a) $\mathfrak{u}(0) \subset \mathfrak{U}$ is a non-empty set called *universe of kinds*;
   (b) $\mathfrak{u}(1) \subset \mathfrak{U}$ is a non-empty set called *universe of properties*;
   (c) $\mathfrak{u}(2) \subset \mathfrak{U}$ is a non-empty set called *universe of objects*;
   (d) $\mathfrak{u}(3) \subset \mathfrak{U}$ is a non-empty set called *universe of tropes*;
   (e) $\mathfrak{u}(4) \subset \mathfrak{U}$ is a non-empty set called *universe of universals*;
   (f) $\mathfrak{u}(5) \subset \mathfrak{U}$ is a non-empty set called *universe of instances*;

(g) $\mathfrak{u}(6) \subset \mathfrak{U}$ is a non-empty set called *universe of substrates*;
(h) $\mathfrak{u}(7) \subset \mathfrak{U}$ is a non-empty set called *universe of attributes*;
(i) $\mathfrak{u}(0) \cup \mathfrak{u}(1)$ is a partition of $\mathfrak{u}(4)$;
(j) $\mathfrak{u}(2) \cup \mathfrak{u}(3)$ is a partition of $\mathfrak{u}(5)$;
(k) $\mathfrak{u}(0) \cup \mathfrak{u}(2)$ is a partition of $\mathfrak{u}(6)$;
(l) $\mathfrak{u}(1) \cup \mathfrak{u}(3)$ is a partition of $\mathfrak{u}(7)$;
(m) $\mathfrak{u}(4) \cup \mathfrak{u}(5)$ is a partition of $\mathfrak{U}$;
(n) $\mathfrak{u}(6) \cup \mathfrak{u}(7)$ is a partition of $\mathfrak{U}$;

3. $\mathfrak{p}$ is a function from the interval $[0, 5]$ into the powerset of $\mathfrak{U} \times \mathfrak{U}$ such that:

   (a) $\mathfrak{p}(0) \subseteq \mathfrak{u}(0) \times \mathfrak{u}(6)$ is antisymmetric and transitive;
   (b) $\mathfrak{p}(1) \subseteq \mathfrak{u}(1) \times \mathfrak{u}(7)$ is antisymmetric and transitive;
   (c) $\mathfrak{p}(2) \subseteq \mathfrak{u}(1) \times \mathfrak{u}(0)$ is such that for any $\mathsf{i} \in \mathfrak{u}(1)$, and any $\mathsf{i}', \mathsf{i}'' \in \mathfrak{u}(0)$, if $\langle \mathsf{i}, \mathsf{i}' \rangle \in \mathfrak{p}(2)$ and $\langle \mathsf{i}', \mathsf{i}'' \rangle \in \mathfrak{p}(1)$, then $\langle \mathsf{i}, \mathsf{i}'' \rangle \in \mathfrak{p}(2)$;
   (d) $\mathfrak{p}(3) \subseteq \mathfrak{u}(3) \times \mathfrak{u}(2)$ is such that for any $\mathsf{i} \in \mathfrak{u}(3)$, there is at most one $\mathsf{i}' \in \mathfrak{u}(2)$, such that $\langle \mathsf{i}, \mathsf{i}' \rangle \in \mathfrak{p}(3)$;
   (e) $\mathfrak{p}(4) \subseteq \mathfrak{u}(1) \times \mathfrak{u}(0)$ is such that for any $\mathsf{i} \in \mathfrak{u}(1)$, and any $\mathsf{i}', \mathsf{i}'' \in \mathfrak{u}(0)$, if $\langle \mathsf{i}, \mathsf{i}' \rangle \in \mathfrak{p}(4)$ and $\langle \mathsf{i}', \mathsf{i}'' \rangle \in \mathfrak{p}(1)$, then $\langle \mathsf{i}, \mathsf{i}'' \rangle \in \mathfrak{p}(4)$;
   (f) $\mathfrak{p}(5) \subseteq \mathfrak{u}(3) \times \mathfrak{u}(2)$, such that for any $\mathsf{i} \in \mathfrak{u}(3)$, there is at most one $\mathsf{i}' \in \mathfrak{u}(2)$, such that $\langle \mathsf{i}, \mathsf{i}' \rangle \in \mathfrak{p}(5)$;

4. $\mathfrak{I}$ is a function called *interpretation* on $\mathfrak{M}$ such that:

   (a) for each kind constant a_k, $\mathfrak{I}(\mathsf{a}_k) \in \mathfrak{u}(0)$;
   (b) for each property constant a_p, $\mathfrak{I}(\mathsf{a}_p) \in \mathfrak{u}(1)$;
   (c) for each object constant a_o, $\mathfrak{I}(\mathsf{a}_o) \in \mathfrak{u}(3)$;
   (d) for each trope constant a_t, $\mathfrak{I}(\mathsf{a}_t) \in \mathfrak{u}(4)$;

5. $\mathfrak{M}$ is assumed to be faithful to the axioms for identity.

An *assignment on* $\mathfrak{M}$ is a function $\mathfrak{A}$ from the set of variables into $\mathfrak{U}$ which assigns to each kind variable an element of $\mathfrak{u}(0)$, to each property variable an element of $\mathfrak{u}(1)$, to each object variable an element of $\mathfrak{u}(2)$, and to each trope variable an element of $\mathfrak{u}(3)$. For any assignments $\mathfrak{A}$, $\mathfrak{A}'$ and any variables x, $\mathfrak{A} \simeq_{\mathsf{x}} \mathfrak{A}'$ means "$\mathfrak{A}$ agrees with $\mathfrak{A}'$ on every variable except possibly x". The *denotation relative to the interpretation* $\mathfrak{I}$ *and the assignment* $\mathfrak{A}$ is the function $\mathfrak{D}_{\mathfrak{I},\mathfrak{A}}$ such that, for any term t, $\mathfrak{D}_{\mathfrak{I},\mathfrak{A}}(\mathsf{t}) = \mathfrak{I}(\mathsf{t})$ iff t is a constant, and $\mathfrak{D}_{\mathfrak{I},\mathfrak{A}}(\mathsf{t}) = \mathfrak{A}(\mathsf{t})$ iff t is a variable.

The *satisfaction* relation between a model $\mathfrak{M}$ for LoOS$^+$, an assignment $\mathfrak{A}$ on $\mathfrak{M}$ and a formula ϕ of LoOS$^+$ ($\mathfrak{M}, \mathfrak{A} \vDash \phi$) can be defined as follows[6]:

$\vDash$ 1 $\mathfrak{M}, \mathfrak{A} \vDash \neg\phi$ iff $\mathfrak{M}, \mathfrak{A} \nvDash \phi$;
$\vDash$ 2 $\mathfrak{M}, \mathfrak{A} \vDash \phi \rightarrow \psi$ iff $\mathfrak{M}, \mathfrak{A} \nvDash \phi$ or $\mathfrak{M}, \mathfrak{A} \vDash \psi$;
$\vDash$ 3 $\mathfrak{M}, \mathfrak{A} \vDash \forall \mathsf{x}\, \phi(\mathsf{x})$ iff for all assignments $\mathfrak{A}'$ such that $\mathfrak{A}' \simeq_{\mathsf{x}} \mathfrak{A}$, $\mathfrak{M}, \mathfrak{A}' \vDash \phi(\mathsf{x})$;
$\vDash$ 4 for any terms t, t', $\mathfrak{M}, \mathfrak{A} \vDash \mathsf{t} = \mathsf{t}'$ iff $\mathfrak{D}_{\mathfrak{I},\mathfrak{A}}(\mathsf{t}) = \mathfrak{D}_{\mathfrak{I},\mathfrak{A}}(\mathsf{t}')$.
$\vDash$ 5 for any terms t, t', $\mathfrak{M}, \mathfrak{A} \vDash \mathsf{t}\,\mathsf{SB}_s\mathsf{t}'$ iff $\langle \mathfrak{D}_{\mathfrak{I},\mathfrak{A}}(\mathsf{t}), \mathfrak{D}_{\mathfrak{I},\mathfrak{A}}(\mathsf{t}') \rangle \in \mathfrak{p}(0)$;
$\vDash$ 6 for any terms t, t', $\mathfrak{M}, \mathfrak{A} \vDash \mathsf{t}\,\mathsf{SB}_a\mathsf{t}'$ iff $\langle \mathfrak{D}_{\mathfrak{I},\mathfrak{A}}(\mathsf{t}), \mathfrak{D}_{\mathfrak{I},\mathfrak{A}}(\mathsf{t}') \rangle \in \mathfrak{p}(1)$;
$\vDash$ 7 for any terms t, t', $\mathfrak{M}, \mathfrak{A} \vDash \mathsf{t}\,\mathsf{AT}_u\mathsf{t}'$ iff $\langle \mathfrak{D}_{\mathfrak{I},\mathfrak{A}}(\mathsf{t}), \mathfrak{D}_{\mathfrak{I},\mathfrak{A}}(\mathsf{t}') \rangle \in \mathfrak{p}(2)$;

[6]Note that copulae are not treated as interpreted "predicates".

$\models 8$ for any terms t, t', $\mathfrak{M}, \mathfrak{A} \models \mathsf{t}\,\mathsf{AT}_i\mathsf{t}'$ iff $\langle \mathfrak{D}_{\mathfrak{I},\mathfrak{A}}(\mathsf{t}), \mathfrak{D}_{\mathfrak{I},\mathfrak{A}}(\mathsf{t}') \rangle \in \mathfrak{p}(3)$;
$\models 9$ for any terms t, t', $\mathfrak{M}, \mathfrak{A} \models \mathsf{t}\,\mathsf{RE}_u\mathsf{t}'$ iff $\langle \mathfrak{D}_{\mathfrak{I},\mathfrak{A}}(\mathsf{t}), \mathfrak{D}_{\mathfrak{I},\mathfrak{A}}(\mathsf{t}') \rangle \in \mathfrak{p}(4)$;
$\models 10$ for any terms t, t', $\mathfrak{M}, \mathfrak{A} \models \mathsf{t}\,\mathsf{RE}_i\mathsf{t}'$ iff $\langle \mathfrak{D}_{\mathfrak{I},\mathfrak{A}}(\mathsf{t}), \mathfrak{D}_{\mathfrak{I},\mathfrak{A}}(\mathsf{t}') \rangle \in \mathfrak{p}(5)$;

A set S of formulae is satisfied by a model $\mathfrak{M}$ and an assignment $\mathfrak{A}$ on $\mathfrak{M}$, iff the latter satisfy every member of S. A formula ϕ is *valid* iff for every model $\mathfrak{M}$ and assignment $\mathfrak{A}$ on $\mathfrak{M}$, $\mathfrak{M}, \mathfrak{A} \models \phi$. An inference rule is *sound* iff every model $\mathfrak{M}$ and assignment $\mathfrak{A}$ on $\mathfrak{M}$ that satisfy its premisses also satisfy its conclusions.

3.2. Soundness

Theorem 1. *Every theorem of LoOS$^+$ is valid.*

Proof. Inspection of the axioms and inference rules of LoOS$^+$ suffices to show its soundness. The logical axioms of LoOS$^+$ are valid and the inference rules of LoOS$^+$ are sound in virtue of clauses $\models 1$ to $\models 3$. Clause $\models 4$ grounds the validity of Axioms 5 and 6, given the assumption that $\mathfrak{M}$ is faithful to the axioms for identity.

Regarding the validity of the axioms of predication, we observe that the conditions 3(a)-(f) on the model-theoretical counterparts of the copulae are direct translations of the axioms. Thus establishing the validity of the axioms of predication is straightforward. Indeed, axioms 7 and 8 are valid in virtue of clause $\models 5$ and subclause 3(a) of the definition of a model for LoOS$^+$. Similarly, Axioms 9 and 10 are valid in virtue of clause $\models 6$ and subclause 3(b) of the definition of a model for LoOS$^+$. Axiom 11 is valid because of clause $\models 7$ and subclause 3(c) of the definition of a model for LoOS$^+$. Clause $\models 8$, together with condition 3(d) of the definition of a model for LoOS$^+$, guarantees the validity of Axiom 12. Axiom 13 is valid because of clause $\models 9$ and subclause 3(e) of the definition of a model for LoOS$^+$. Axiom 14 is valid in virtue of clause $\models 10$, together with subclause 3(f) of the definition of a model for LoOS$^+$. □

3.3. Completeness

First some definitions. A set S of formulae is *AX-consistent* (i.e. axiom-consistent) if, and only if there is no formula ϕ such that $S \vdash_{LoOS^+} \neg(\phi \to \phi)$ (indeed, $\phi \to \phi$ is a theorem of LoOS$^+$)[7]. A set of formulae S is *maximally AX-consistent* if, and only if S is *AX*-consistent and for every formula ϕ, either $\phi \in S$ or $\neg\phi \in S$. A set of LoOS$^+$-formulae S is *normal*, iff for any sort i, if $\exists \mathsf{x}_i\, \phi(\mathsf{x}_i) \in S$, then there is at least one constant a_i such that $\phi(\mathsf{a}_i) \in S$.

Theorem 2. *Every valid formula of LoOS$^+$ is a theorem of LoOS$^+$.*

Since the proof runs as as for the completeness of LoOS, it suffices to establish Lemma 1 and Lemma 2 below.

Lemma 1. *Each AX-consistent set is a subset of a maximally AX-consistent normal set.*

[7]The notion of derivability expressed by "$\vdash_{LoOS^+}$" involves not only the logical axioms and inference rules, but also the axioms of predication.

Proof. The assumption is that the signature of LoOS$^+$ contains denumerably many constants for each sort; otherwise it should be extended in such a manner that it is. The set of LoOS$^+$-constants will serve as a set of so-called *witnesses*. Let us assume that the formulae of extended LoOS$^+$, as well as its constants, are enumerated separately. Given any AX-consistent set of formulae S, one constructs a cumulative sequence of sets S_i in the following way:

> $S_0 = S$
> $S_{n+1} = S_n \cup \{\exists \mathsf{x}_j\, \phi(\mathsf{x}_j) \rightarrow \phi(\mathsf{a}_j)\}$ if the n+1-th formula is of the form $\exists \mathsf{x}_j\, \phi(\mathsf{x}_j)$, a_j being the first constant of sort j that does not appear either in S_n or in $\exists \mathsf{x}_j\, \phi(\mathsf{x}_j)$;
> otherwise: $S_{n+1} = S_n$

Let S' be the union of all S_i: by induction it can be shown that every S_i is AX-consistent, and hence also S'. S' can be extended to a both normal and maximally AX-consistent set S^* as follows (enumerating anew the formulae not used up in the previous steps):

> $S^*_0 = S'$
> $S^*_{n+1} = S^*_n \cup \{\phi\}$ if the n+1-th formula ϕ is such that $S_n \cup \{\phi\}$ is AX-consistent; otherwise:
> $S^*_{n+1} = S^*_n$

S^* is the union of the S^*_i. □

Now, it can be shown that, since S^* is a maximally AX-consistent set, the following statements hold:

S*1 for all terms t, t', if $\ulcorner \mathsf{t}\,\mathsf{SB}_s\mathsf{t}' \urcorner \in S^*$ and $\ulcorner \mathsf{t}'\mathsf{SB}_s\mathsf{t} \urcorner \in S^*$, then $\ulcorner \mathsf{t} = \mathsf{t}' \urcorner \in S^*$;
S*2 for all terms $\mathsf{t}, \mathsf{t}', \mathsf{t}''$, if $\ulcorner \mathsf{t}\mathsf{SB}_s\mathsf{t}' \urcorner \in S^*$ and $\ulcorner \mathsf{t}'\mathsf{SB}_s\mathsf{t}'' \urcorner \in S^*$, then $\ulcorner \mathsf{t}\mathsf{SB}_s\mathsf{t}'' \urcorner \in S^*$;
S*3 for all terms t, t', if $\ulcorner \mathsf{t}\,\mathsf{SB}_a\mathsf{t}' \urcorner \in S^*$ and $\ulcorner \mathsf{t}'\mathsf{SB}_a\mathsf{t} \urcorner \in S^*$, then $\ulcorner \mathsf{t} = \mathsf{t}' \urcorner \in S^*$;
S*4 for all terms $\mathsf{t}, \mathsf{t}', \mathsf{t}''$, if $\ulcorner \mathsf{t}\,\mathsf{SB}_a\mathsf{t}' \urcorner \in S^*$ and $\ulcorner \mathsf{t}'\mathsf{SB}_a\mathsf{t}'' \urcorner \in S^*$, then $\ulcorner \mathsf{t}\,\mathsf{SB}_a\mathsf{t}'' \urcorner \in S^*$;
S*5 for all terms $\mathsf{t}, \mathsf{t}', \mathsf{t}''$, if $\ulcorner \mathsf{t}\,\mathsf{AT}_u\mathsf{t}' \urcorner \in S^*$ and $\ulcorner \mathsf{t}'\mathsf{SB}_s\mathsf{t}'' \urcorner \in S^*$, then $\ulcorner \mathsf{t}\,\mathsf{AT}_u\mathsf{t}'' \urcorner \in S^*$;
S*6 for all terms $\mathsf{t}, \mathsf{t}', \mathsf{t}''$, if $\ulcorner \mathsf{t}\,\mathsf{AT}_i\mathsf{t}' \urcorner \in S^*$ and $\ulcorner \mathsf{t}\,\mathsf{AT}_i\mathsf{t}'' \urcorner \in S^*$, then $\ulcorner \mathsf{t}' = \mathsf{t}'' \urcorner \in S^*$;
S*7 for all terms $\mathsf{t}, \mathsf{t}', \mathsf{t}''$, if $\ulcorner \mathsf{t}\,\mathsf{RE}_u\mathsf{t}' \urcorner \in S^*$ and $\ulcorner \mathsf{t}'\mathsf{SB}_s\mathsf{t}'' \urcorner \in S^*$, then $\ulcorner \mathsf{t}\,\mathsf{RE}_u\mathsf{t}'' \urcorner \in S^*$;
S*8 for all terms $\mathsf{t}, \mathsf{t}', \mathsf{t}''$, if $\ulcorner \mathsf{t}\,\mathsf{RE}_i\mathsf{t}' \urcorner \in S^*$ and $\ulcorner \mathsf{t}\,\mathsf{RE}_i\mathsf{t}'' \urcorner \in S^*$, then $\ulcorner \mathsf{t}' = \mathsf{t}'' \urcorner \in S^*$;

Proof. We first note that S* is deductively closed, i.e.

DC S$^* \vdash_{LoOS^+} \phi$ iff $\ulcorner \phi \urcorner \in$S*.

Indeed, if S$^* \vdash_{LoOS^+} \phi$, then S$^* \nvdash_{LoOS^+} \neg\phi$ (by consistency), hence $\ulcorner \neg\phi \urcorner \notin$S*, and thus $\ulcorner \phi \urcorner \in$S* (by maximality).

We then observe that S*1 and S*3 are of the form (1a), S*2, S*4, S*5 and S*7 are of the form (1b), and S*6 as well as S*8 are of the form (1c):

(1a)	for any t, t', if $\ulcorner \phi(\mathsf{t}, \mathsf{t}') \urcorner \in S^*$ and $\ulcorner \psi(\mathsf{t}', \mathsf{t}) \urcorner \in S^*$, then $\ulcorner \xi(\mathsf{t}, \mathsf{t}') \urcorner \in S^*$
(1b)	for any $\mathsf{t}, \mathsf{t}', \mathsf{t}''$, if $\ulcorner \phi(\mathsf{t}, \mathsf{t}') \urcorner \in S^*$ and $\ulcorner \psi(\mathsf{t}', \mathsf{t}'') \urcorner \in S^*$, then $\ulcorner \xi(\mathsf{t}, \mathsf{t}'') \urcorner \in S^*$
(1c)	for any $\mathsf{t}, \mathsf{t}', \mathsf{t}''$, if $\ulcorner \phi(\mathsf{t}, \mathsf{t}') \urcorner \in S^*$ and $\ulcorner \psi(\mathsf{t}, \mathsf{t}'') \urcorner \in S^*$, then $\ulcorner \xi(\mathsf{t}', \mathsf{t}'') \urcorner \in S^*$

The respective conditionals, namely:

(i) $\ulcorner(\,\phi(\mathsf{t},\mathsf{t}') \wedge \psi(\mathsf{t}',\mathsf{t})\,) \rightarrow \xi(\mathsf{t},\mathsf{t}')\urcorner$
(ii) $\ulcorner(\,\phi(\mathsf{t},\mathsf{t}') \wedge \psi(\mathsf{t}',\mathsf{t}'')\,) \rightarrow \xi(\mathsf{t},\mathsf{t}'')\urcorner$
(iii) $\ulcorner(\,\phi(\mathsf{t},\mathsf{t}') \wedge \psi(\mathsf{t},\mathsf{t}'')\,) \rightarrow \xi(\mathsf{t}',\mathsf{t}'')\urcorner$

are such that instances of (i) follow from axiom 7 respectively 9, instances of (ii) follow from axiom 8, 10, 11 or 13 and instances of (iii) follow from axiom 12 or 14, by eliminating all quantifiers through successively using axiom 4 and rule 1.

Now, in virtue of **DC** (the deductive closure of S^*), all the axioms are in S^*. Hence all instances of the form (i), (ii) or (iii) are LoOS*-derivable from S^*, and thus, by **DC**:

(2a) for any t,t', $\ulcorner(\,\phi(\mathsf{t},\mathsf{t}') \wedge \psi(\mathsf{t}',\mathsf{t})\,) \rightarrow \xi(\mathsf{t},\mathsf{t}')\urcorner \in S^*$
(2b) for any $\mathsf{t},\mathsf{t}',\mathsf{t}''$, $\ulcorner(\,\phi(\mathsf{t},\mathsf{t}') \wedge \psi(\mathsf{t}',\mathsf{t}'')\,) \rightarrow \xi(\mathsf{t},\mathsf{t}'')\urcorner \in S^*$
(2c) for any $\mathsf{t},\mathsf{t}',\mathsf{t}''$, $\ulcorner(\,\phi(\mathsf{t},\mathsf{t}') \wedge \psi(\mathsf{t},\mathsf{t}'')\,) \rightarrow \xi(\mathsf{t}',\mathsf{t}'')\urcorner \in S^*$

Now suppose, by hypothesis, that

(3a) $\ulcorner\phi(\mathsf{t},\mathsf{t}')\urcorner \in S^*$ and $\ulcorner\psi(\mathsf{t}',\mathsf{t})\urcorner \in S^*$
(3b) $\ulcorner\phi(\mathsf{t},\mathsf{t}')\urcorner \in S^*$ and $\ulcorner\psi(\mathsf{t}',\mathsf{t}'')\urcorner \in S^*$
(3c) $\ulcorner\phi(\mathsf{t},\mathsf{t}')\urcorner \in S^*$ and $\ulcorner\psi(\mathsf{t},\mathsf{t}'')\urcorner \in S^*$

In virtue of **DC**, the respective conjunctions will be in S^*, hence:

(4a) $\ulcorner\phi(\mathsf{t},\mathsf{t}') \wedge \psi(\mathsf{t}',\mathsf{t})\urcorner \in S^*$
(4b) $\ulcorner\phi(\mathsf{t},\mathsf{t}') \wedge \psi(\mathsf{t}',\mathsf{t}'')\urcorner \in S^*$
(4c) $\ulcorner\phi(\mathsf{t},\mathsf{t}') \wedge \psi(\mathsf{t},\mathsf{t}'')\urcorner \in S^*$

Now, again by **DC**, (2a) and (4a) imply (5a), (2b) and (4b) imply (5b), while (2c) and (4c) imply (5c):

(5a) $\ulcorner\xi(\mathsf{t},\mathsf{t}')\urcorner \in S^*$
(5b) $\ulcorner\xi(\mathsf{t},\mathsf{t}'')\urcorner \in S^*$
(5c) $\ulcorner\xi(\mathsf{t}',\mathsf{t}'')\urcorner \in S^*$

which concludes the proof of S^*1 to S^*8. □

Lemma 2. *Every maximally AX-consistent normal set has a model.*

Proof. Let S^* be a maximally AX-consistent set constructed as described above. A model $\mathfrak{M}^* = \langle \mathfrak{U}^*_s, \mathfrak{u}^*, \mathfrak{p}^*, \mathfrak{I}^* \rangle$ for S^* can be devised in the following manner.

The *identity class* of a term t is the equivalence class of terms t' such that $\ulcorner \mathsf{t} = \mathsf{t}' \urcorner \in S^*$. Let then $\mathfrak{u}^*(0)$ be the set of identity classes of kind terms, $\mathfrak{u}^*(1)$ the set of identity classes of property terms, $\mathfrak{u}^*(2)$ the set of identity classes of object terms, and $\mathfrak{u}^*(3)$ the set of identity classes of trope terms; $\mathfrak{u}^*(4)$ to $\mathfrak{u}^*(7)$, as well as $\mathfrak{U}$ are then constructed bottom-up from the previous sets as required. The interpretation $\mathfrak{I}^*$ of a constant a is its identity class; similarly the assignment $\mathfrak{A}^*$ for a variable x is the identity class of x. The definition of the denotation $\mathfrak{D}^*_{\mathfrak{I}^*,\mathfrak{A}^*}$ relative to $\mathfrak{I}^*$ and $\mathfrak{A}^*$ is straightforward.

Furthermore, for any terms t,t', I stipulate:

1. $\langle \mathfrak{D}^*_{\mathfrak{I}^*,\mathfrak{A}^*}(\mathsf{t}), \mathfrak{D}^*_{\mathfrak{I}^*,\mathfrak{A}^*}(\mathsf{t}') \rangle \in \mathfrak{p}^*(0)$ iff $\ulcorner \mathsf{t}\,\mathsf{SB}_s\mathsf{t}' \urcorner \in S^*$; by S^*1 and S^*2; $\mathfrak{p}^*(0)$ satisfies condition 3(a) in the definition of a model of LoOS$^+$;
2. $\langle \mathfrak{D}^*_{\mathfrak{I}^*,\mathfrak{A}^*}(\mathsf{t}), \mathfrak{D}^*_{\mathfrak{I}^*,\mathfrak{A}^*}(\mathsf{t}') \rangle \in \mathfrak{p}^*(1)$ iff $\ulcorner \mathsf{t}\,\mathsf{SB}_a\mathsf{t}' \urcorner \in S^*$; by S^*3 and S^*4, $\mathfrak{p}^*(1)$ satisfies condition 3(b) in the definition of a model of LoOS$^+$;

3. $\langle \mathfrak{D}^*_{\mathfrak{I}^*,\mathfrak{A}^*}(\mathsf{t}), \mathfrak{D}^*_{\mathfrak{I}^*,\mathfrak{A}^*}(\mathsf{t}') \rangle \in \mathfrak{p}^*(2)$ iff $\ulcorner \mathsf{t}\,\mathsf{AT}_u \mathsf{t}' \urcorner \in S^*$; by S*5, $\mathfrak{p}^*(2)$ satisfies condition 3(c) in the definition of a model of LoOS$^+$;
4. $\langle \mathfrak{D}^*_{\mathfrak{I}^*,\mathfrak{A}^*}(\mathsf{t}), \mathfrak{D}^*_{\mathfrak{I}^*,\mathfrak{A}^*}(\mathsf{t}') \rangle \in \mathfrak{p}^*(3)$ iff $\ulcorner \mathsf{t}\,\mathsf{AT}_i \mathsf{t}' \urcorner \in S^*$; by S*6, $\mathfrak{p}^*(3)$ satisfies condition 3(d) in the definition of a model of LoOS$^+$;
5. $\langle \mathfrak{D}^*_{\mathfrak{I}^*,\mathfrak{A}^*}(\mathsf{t}), \mathfrak{D}^*_{\mathfrak{I}^*,\mathfrak{A}^*}(\mathsf{t}') \rangle \in \mathfrak{p}^*(4)$ iff $\ulcorner \mathsf{t}\,\mathsf{RE}_u \mathsf{t}' \urcorner \in S^*$; by S*7, $\mathfrak{p}^*(4)$ satisfies condition 3(e) in the definition of a model of LoOS$^+$;
6. $\langle \mathfrak{D}^*_{\mathfrak{I}^*,\mathfrak{A}^*}(\mathsf{t}), \mathfrak{D}^*_{\mathfrak{I}^*,\mathfrak{A}^*}(\mathsf{t}') \rangle \in \mathfrak{p}^*(5)$ iff $\ulcorner \mathsf{t}\,\mathsf{RE}_i \mathsf{t}' \urcorner \in S^*$; by S*8, $\mathfrak{p}^*(5)$ satisfies condition 3(f) in the definition of a model of LoOS$^+$.

That $\mathfrak{M}^*$ together with assignment $\mathfrak{A}^*$ satisfies S^* can be shown by induction on the degree of the formulae in S^*. More precisely, what one needs to show is that for every ϕ, $\mathfrak{M}^*, \mathfrak{A}^* \vDash \phi$ iff $\phi \in S^*$. This has first to be established for atomic formulae, where seven subcases have to be distinguished:

1. ϕ is of the form $\ulcorner \mathsf{t} = \mathsf{t}' \urcorner$: by construction of $\mathfrak{D}^*_{\mathfrak{I}^*,\mathfrak{A}^*}$, $\mathfrak{D}^*_{\mathfrak{I}^*,\mathfrak{A}^*}(\mathsf{t}) = \mathfrak{D}^*_{\mathfrak{I}^*,\mathfrak{A}^*}(\mathsf{t}')$ iff $\ulcorner \mathsf{t} = \mathsf{t}' \urcorner \in S^*$. Therefore, by $\vDash 4$, $\mathfrak{M}^*, \mathfrak{A}^* \vDash \mathsf{t} = \mathsf{t}'$ iff $\ulcorner \mathsf{t} = \mathsf{t}' \urcorner \in S^*$.
2. ϕ is of the form $\ulcorner \mathsf{t}\,\mathsf{SB}_s \mathsf{t}' \urcorner$: by construction of $\mathfrak{p}^*(0)$, $\langle \mathfrak{D}^*_{\mathfrak{I}^*,\mathfrak{A}^*}(\mathsf{t}), \mathfrak{D}^*_{\mathfrak{I}^*,\mathfrak{A}^*}(\mathsf{t}') \rangle \in \mathfrak{p}^*(0)$ iff $\ulcorner \mathsf{t}\,\mathsf{SB}_s \mathsf{t}' \urcorner \in S^*$. Thus, by $\vDash 5$, $\mathfrak{M}^*, \mathfrak{A}^* \vDash \mathsf{t}\,\mathsf{SB}_s \mathsf{t}'$ iff $\ulcorner \mathsf{t}\,\mathsf{SB}_s \mathsf{t}' \urcorner \in S^*$.
3. ϕ is of the form $\ulcorner \mathsf{t}\,\mathsf{SB}_a \mathsf{t}' \urcorner$: by construction of $\mathfrak{p}^*(1)$, $\langle \mathfrak{D}^*_{\mathfrak{I}^*,\mathfrak{A}^*}(\mathsf{t}), \mathfrak{D}^*_{\mathfrak{I}^*,\mathfrak{A}^*}(\mathsf{t}') \rangle \in \mathfrak{p}^*(1)$ iff $\ulcorner \mathsf{t}\,\mathsf{SB}_a \mathsf{t}' \urcorner \in S^*$. Thus, by $\vDash 6$, $\mathfrak{M}^*, \mathfrak{A}^* \vDash \mathsf{t}\,\mathsf{SB}_a \mathsf{t}'$ iff $\ulcorner \mathsf{t}\,\mathsf{SB}_a \mathsf{t}' \urcorner \in S^*$.
4. ϕ is of the form $\ulcorner \mathsf{t}\,\mathsf{AT}_u \mathsf{t}' \urcorner$: by construction of $\mathfrak{p}^*(2)$, $\langle \mathfrak{D}^*_{\mathfrak{I}^*,\mathfrak{A}^*}(\mathsf{t}), \mathfrak{D}^*_{\mathfrak{I}^*,\mathfrak{A}^*}(\mathsf{t}') \rangle \in \mathfrak{p}^*(2)$ iff $\ulcorner \mathsf{t}\,\mathsf{AT}_u \mathsf{t}' \urcorner \in S^*$. Thus, by $\vDash 7$, $\mathfrak{M}^*, \mathfrak{A}^* \vDash \mathsf{t}\,\mathsf{AT}_u \mathsf{t}'$ iff $\ulcorner \mathsf{t}\,\mathsf{AT}_u \mathsf{t}' \urcorner \in S^*$.
5. ϕ is of the form $\ulcorner \mathsf{t}\,\mathsf{AT}_i \mathsf{t}' \urcorner$: by construction of $\mathfrak{p}^*(3)$, $\langle \mathfrak{D}^*_{\mathfrak{I}^*,\mathfrak{A}^*}(\mathsf{t}), \mathfrak{D}^*_{\mathfrak{I}^*,\mathfrak{A}^*}(\mathsf{t}') \rangle \in \mathfrak{p}^*(3)$ iff $\ulcorner \mathsf{t}\,\mathsf{AT}_i \mathsf{t}' \urcorner \in S^*$. Thus, by $\vDash 8$, $\mathfrak{M}^*, \mathfrak{A}^* \vDash \mathsf{t}\,\mathsf{AT}_i \mathsf{t}'$ iff $\ulcorner \mathsf{t}\,\mathsf{AT}_i \mathsf{t}' \urcorner \in S^*$.
6. ϕ is of the form $\ulcorner \mathsf{t}\,\mathsf{RE}_u \mathsf{t}' \urcorner$: by construction of $\mathfrak{p}^*(4)$, $\langle \mathfrak{D}^*_{\mathfrak{I}^*,\mathfrak{A}^*}(\mathsf{t}), \mathfrak{D}^*_{\mathfrak{I}^*,\mathfrak{A}^*}(\mathsf{t}') \rangle \in \mathfrak{p}^*(2)$ iff $\ulcorner \mathsf{t}\,\mathsf{RE}_u \mathsf{t}' \urcorner \in S^*$. Thus, by $\vDash 9$, $\mathfrak{M}^*, \mathfrak{A}^* \vDash \mathsf{t}\,\mathsf{RE}_u \mathsf{t}'$ iff $\ulcorner \mathsf{t}\,\mathsf{RE}_u \mathsf{t}' \urcorner \in S^*$.
7. ϕ is of the form $\ulcorner \mathsf{t}\,\mathsf{RE}_i \mathsf{t}' \urcorner$: by construction of $\mathfrak{p}^*(5)$, $\langle \mathfrak{D}^*_{\mathfrak{I}^*,\mathfrak{A}^*}(\mathsf{t}), \mathfrak{D}^*_{\mathfrak{I}^*,\mathfrak{A}^*}(\mathsf{t}') \rangle \in \mathfrak{p}^*(5)$ iff $\ulcorner \mathsf{t}\,\mathsf{RE}_i \mathsf{t}' \urcorner \in S^*$. Thus, by $\vDash 10$, $\mathfrak{M}^*, \mathfrak{A}^* \vDash \mathsf{t}\,\mathsf{RE}_i \mathsf{t}'$ iff $\ulcorner \mathsf{t}\,\mathsf{RE}_i \mathsf{t}' \urcorner \in S^*$.

For the inductive step, assume that it has been shown that $\mathfrak{M}^*$ is a model for all formulae of S^* up to, but not including, degree n. Formulae of degree n will be of one of the forms $\neg\phi$, $\phi \wedge \psi$, $\forall \mathsf{v}(\phi)$. The subproofs for each of these three cases runs as for a calculus of standard (many-sorted) predicate logic. In particular, as shown above, axioms 7 to 14 are contained in S^* in virtue of the deductive closure of S*, and, since $\mathfrak{M}^*$ by construction is faithful to these axioms, they will also be true in $\mathfrak{M}^*$. $\square$

Acknowledgements. The author would like to thank three anonymous referees for their comments and suggestions.

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